



## **Thinking Process of Secondary Level Students in Constructing Proof by Mathematical Induction in Terms of Their Attitude toward Mathematics**

**Nurul Laili<sup>1</sup>, Tatag Yuli Eko Siswono<sup>2</sup>**

<sup>1,2</sup>*Pendidikan Matematika, Universitas Negeri Surabaya.*

*Jl. Ketintang, Ketintang, Kec. Gayungan, Kota Surabaya, Jawa Timur 60231*

*e-mail: nurul.17030174057@mhs.unesa.ac.id<sup>1</sup>, tatagsiswono@unesa.ac.id<sup>2</sup>*

### **ABSTRAK**

Penelitian ini bertujuan untuk mendeskripsikan proses berpikir siswa SMA yang ditinjau dari sikap terhadap matematika dalam menyusun pembuktian dengan induksi matematika. Penelitian kualitatif ini melibatkan dua siswa yang dipilih dari 30 siswa kelas 11 SMA yang dikategorikan menjadi dua kelompok, yaitu siswa yang memiliki sikap positif dan negatif terhadap matematika dengan menggunakan angket *Attitude Toward Mathematics Inventory* (ATMI). Satu siswa dari setiap kategori dipilih untuk diberikan tes pembuktian dan diwawancarai. Hasil tes pembuktian dan rekaman wawancara dikumpulkan dan dianalisis untuk mengidentifikasi proses berpikir mereka. Penelitian ini menunjukkan bahwa baik siswa yang memiliki sikap positif maupun negatif terhadap matematika bisa memanggil informasi yang ada di memorinya mengenai ciri-ciri serta langkah-langkah membuktikan menggunakan induksi matematika. Akan tetapi, siswa yang memiliki sikap negatif terhadap matematika cenderung mengalami kesulitan dalam memproses informasi, terutama dalam tahap langkah induksi yang disebabkan oleh rasa panik, tertekan, dan rasa tidak percaya diri. Siswa yang memiliki sikap positif terhadap matematika juga mengalami beberapa kesulitan dalam poses pembuktiannya, yaitu pada saat memahami apa yang harus dibuktikan dan saat menyusun langkah induksi. Akan tetapi, ia terus mencoba dan percaya bahwa dia pasti bisa menyelesaikannya. Sehingga, ia pun dapat menyelesaikan pembuktian dengan lengkap.

**Kata Kunci:** Induksi Matematika, Pembuktian, Proses Berpikir, Sikap terhadap Matematika

### **ABSTRACT**

*This research aims to describe secondary-level students' thinking processes in terms of their attitude toward mathematics in constructing proof by mathematical induction. This qualitative research involved two students who were selected from 30 students of second grade of senior high school that categorized into two groups, namely the students who have a positive and negative attitude toward mathematics by using Attitude Toward Mathematics Inventory (ATMI) questionnaire. One student from each category was selected to be given a proving test and interviewed. Students' process of proving test and interview recording were collected and analyzed to identify their thinking process. This research points out that both students who have positive and negative attitudes towards mathematics can recall the information in their memory regarding the characteristics and steps to prove using mathematical induction. However, the student who has negative attitudes towards mathematics tends to experience some difficulties in processing information, especially in the induction step, caused by panic, depression, and insecurity. The student who has a positive attitude towards mathematics also experiences some difficulties in the proving process, namely understanding what to prove and compiling the induction steps. However, she keeps trying and believes that she would be able to solve it. Hence, she can solve the proof completely.*

**Keywords:** *Mathematical Induction, Proof, Thinking Process, Attitude toward Mathematics*

## INTRODUCTION

A process to show the truth of a mathematical proposition deductively using a valid argument is called mathematical proving (Doruk, 2019). It is recognized as an essential aspect of mathematics education because it involves deductive and logical reasoning that increases students' comprehension in mathematics, critical thinking, and argumentation (Cyr, 2011; Hemmi & Löfwall, 2010; Warli, Cintamulya, & Rahayu, 2020). It is also able to improve problem-solving and mathematical communication skills. It can show who understands mathematics deeply because it involves high-order thinking skills in building ideas and expressing them logically and systematically to construct the proof (Güler, 2016; Pantaleon, Juniati, Lukito, & Mandur, 2018).

Mathematical proof consists of direct and indirect proof methods and mathematical induction (Adinata, Budiyo, & Indriati, 2020). Mathematical induction is considered one of the most powerful tools for proving some mathematical statements or expressions regarding discrete mathematics (Ahmadi, Kusumah, & Jupri, 2019; Ashkenazi & Itzkovitch, 2014). It is a mathematical proof technique to prove formally that a particular property holds for all natural numbers. It starts to be taught at a senior secondary level that is usually used to prove a mathematical statement in sequence, inequality, and division (Adinata et al., 2020; Relaford-doyle, 2020). It consists of two steps: the base and the induction step. The base step shows that the property holds for some initial value, typically  $n = 1$ . Then, the inductive step shows that if the property holds for some arbitrary value  $k$ , it also holds for its successor,  $k + 1$ . The Axiom of Induction (one of the Dedekind-Peano axioms of natural number) follows that the property is valid for all natural numbers (Relaford-doyle, 2020).

To implement mathematical induction to prove a particular claim, students have to carry out some steps successfully. They are identifying the predicate  $P(n)$ , establishing the base step, stating and distinguishing between  $P(k)$  and  $P(k + 1)$ , and the last one is establishing the inductive step from  $P(k)$  to  $P(k + 1)$  (Cooper, 2019). The core of deduction in the proof lies in establishing the induction step (Relaford-doyle, 2020).

Mathematical induction material is essential to be studied by students because it can develop inductive and deductive conclusions (Adinata et al., 2020). For the importance of mathematical induction in mathematics education, there is a lot of research exists about this topic, but there is nearly no research about how the thinking process of secondary students in proving by mathematical induction. Adinata et al. (2020), Ahmadi et al. (2019), Ashkenazi and Itzkovitch (2014), and Hasan (2016) are some researchers that conducted studies related to mathematical induction.

The thinking process has an important role in mathematical activity, such as mathematical problem solving and mathematical proving (Widyastuti, 2015). It is a mental activity that occurs in one's mind to process the information they received (Isroil, Budayasa, & Masriyah, 2017; Mudhiah & Amin, 2020). Mudhiah and Amin (2020) explain that the thinking process involves four stages:

(1) receiving information, that is obtaining specific information from the environment for further processing; (2) processing information, which is an effort to link previous knowledge with the information received; (3) storing information, that is maintaining information received and retaining previous knowledge in memory; and (4) recalling information, that is remembering information received and remembering previous knowledge.

In doing mathematics, everyone has a different thinking process because they have different ways of processing information (Ngilawajan, 2013). Some factors cause the difference of one's thinking process in doing mathematics from another. One of them is a cognitive factor, which is considered as the main factor of one's thinking process. Some mathematicians believe that mathematical activity has strong interaction between cognitive and emotional aspects, such as their attitude toward mathematics (Martino & Zan, 1992). A positive attitude toward mathematics is connected to students' success in mathematics which the success is never separated from their thinking process in doing mathematics (Martino & Zan, 1992). In this way, we can see the relationship between attitude toward mathematics to the thinking process because the thinking process also affected by emotional aspects such as enjoyment of mathematics, motivation to do mathematics, self-confidence in mathematics, and perceived value of mathematics, which are part of attitude toward mathematics (Lim & Chapman, 2013; Tapia, 1996).

This study is discussing about the thinking process of secondary level students in constructing proof by mathematical induction in terms of their attitude toward mathematics. Hasan (2016) discussed the thinking process of university students in constructing mathematical induction proof. However, it used a different thinking process theory with the one in this study, involved university student as its subjects, and did not discuss about attitude toward mathematics as in this study. Isroil et al. (2017) discussed the thinking process of junior high school students on problem solving in terms of mathematical abilities. Mudhiah and Amin (2020) also discussed the same topic but different point of view. The thinking process used in both study is the same as the one in this study, but they discussed problem solving while this research discussed about proving. Apart from that, the research subjects are also different.

There are also some relevant researches on attitude toward mathematics, but most of them are quantitative research, whereas according to Martino and Zan (1992) attitude is a qualitative thing and needs to be studied in depth because there are many aspects of attitude that cannot be quantified. The position of this research among the researches that have been done is to adapt the theory of thinking process to construct thinking process indicator and adopt questionnaire to classify students' attitude toward mathematics, then qualitatively examine how the role of the attitude in students' thinking processes when constructing mathematical induction proof because there are still very few studies that discuss this.

From the background, it is necessary to study the thinking process of secondary level students in constructing proof by mathematical induction in terms of student's attitudes toward

mathematics. This research aims to describe secondary-level students' thinking processes with various attitudes toward mathematics in constructing proof by mathematical induction.

## METHOD

This case study involved two students that selected from 30 students in one of private senior high schools in Pamekasan, East Java. They were grouped based on their attitude toward mathematics, positive and negative, known through the Questionnaire of Attitude Toward Mathematics Inventory (ATMI) adopted from [Lim and Chapman \(2013\)](#). The questionnaire consisted of 40 items that measure four aspects: enjoyment of mathematics (ENJ), motivation to do mathematics (MOT), self-confidence in mathematics (SC), and perceived value of mathematics (VAL) which containing 10, 5, 15, and 10 items, respectively ([Lim & Chapman, 2013](#)). Sample statements of each aspect are displayed in [Table 1](#).

**Table 1.** Sample Statements of the Attitude Aspects

Attitude aspects	Sample Statements
Enjoyment	<ul style="list-style-type: none"> <li>• I get a great deal of satisfaction out of solving a mathematics problem.</li> <li>• I have usually enjoyed studying mathematics in school.</li> <li>• Mathematics is dull and boring.*</li> </ul>
Motivation	<ul style="list-style-type: none"> <li>• I would like to avoid using mathematics in university.*.</li> <li>• I plan to take as much mathematics as I can during my education.</li> <li>• The challenge of mathematics appeals to me</li> </ul>
Self-confidence	<ul style="list-style-type: none"> <li>• My mind goes blank and I am unable to think clearly when working with mathematics.*</li> <li>• I am able to solve mathematics problems without too much difficulty</li> <li>• I believe I am good at solving mathematics problems.</li> </ul>
Value	<ul style="list-style-type: none"> <li>• Mathematics is a very worthwhile and necessary subject.</li> <li>• I believe studying mathematics helps me with problem-solving in other areas.</li> <li>• A strong mathematics background could help me in my professional life.</li> </ul>

Note: (\*) unfavourable item

All statements composing attitude aspects were scored on a 5-point scale ranging from 5 = strongly agree to 1 = strongly disagree. At this point, negative items were scored in the reverse order. The student with a negative attitude toward mathematics is the one who has a total score of 40-120, and the positive one is the one who has a total score of 121-200. A student from each category was selected and they were given a proving test as the representatives. The consideration in choosing the two subjects is based on their ability in mathematics and their gender. Ability in mathematics and gender may affect the thinking process of the students. So, we control them because we want to focus on how attitude toward mathematics affect their thinking process in constructing mathematical induction proofs. By having consultation with their mathematics teacher, we selected those who are good in mathematics, having relatively the same ability, and having good communication (to make it easier to collect data during the interview). We selected

those who are good in mathematics because proving needs good mathematics skill. The given proof task consisted of two problems as follow:

1. Prove that  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ , for all  $n$  element of natural number.
2. Prove that the sum of  $n$  consecutive cubic natural number is equal to the square of the sum of  $n$  consecutive natural number.

The two problems have different levels of difficulty. The first problem is presented in a mathematical expression while the second one is presented in a sentence. It means that the second problem requires more understanding in order to be able to present it in a mathematical expression. It is also more complex than the first one. It requires more comprehension and carefulness in constructing the induction step and relating it to the first problem because it has a relationship with the first one. The two problems have been consulted with an expert related to the construction of the problems, materials, level of difficulties, and language in the problems.

After the research subjects finished the proving test, then we did the recorded interviews. The interview data were analyzed qualitatively by making a transcript, reducing data, displaying data, and finally drawing conclusions. The interview data were analyzed based on the indicator of thinking process that adapted from research of Fauziyah (2020), Isroil et al. (2017), and Mudhiah & Amin (2020). Even though the thinking process indicators were used in problem-solving, but we use them in proving because Furinghetti and Morselli (2009) made a connection between proof and problem-solving by considering proof as a particular case of problem-solving, and even Hemmi and Löfwall (2010) argued that these two activities are the same. So, we adapted their thinking process indicators to be used in this study. The thinking process indicators were also used as interview guidelines in obtaining information about how the students receive, store, recall, and process information in every stage (understanding, planning, carrying out, and looking back). We also asked their feeling when they working with the proof problems.

## RESULT AND DISCUSSION

Based on the ATMI Questionnaire, we found that there are 9 students with a positive attitude and 21 students with a negative attitude toward mathematics. From the data, we selected one student from each category with the same gender, having good mathematical ability, and good communication. By having consultation with the teacher, we selected them based on the criteria, and we get one female student with positive attitude toward mathematics (S1) and one female student with negative attitude toward mathematics (S2). They were given a proving test and interviewed to know their thinking process in proving by mathematical induction. The data of their thinking process are as follows.

### Thinking Process of Student with Positive Attitude toward Mathematics (S1)

On the first problem, S1 began the proof by understanding what she needs to prove. At this stage, she received information by reading the task sheet and stored the information by reading the task and paraphrasing in her language what she had to prove. Then she recalled information in her memory if she had encountered a similar problem. She stated that she had ever encountered a similar problem, namely proving the series formula in the chapter on mathematical induction. Then she processed information by associating the information in the sheet task with the information she remembered. Finally, she understood that what she needs to do was proving the formula in the sheet task.

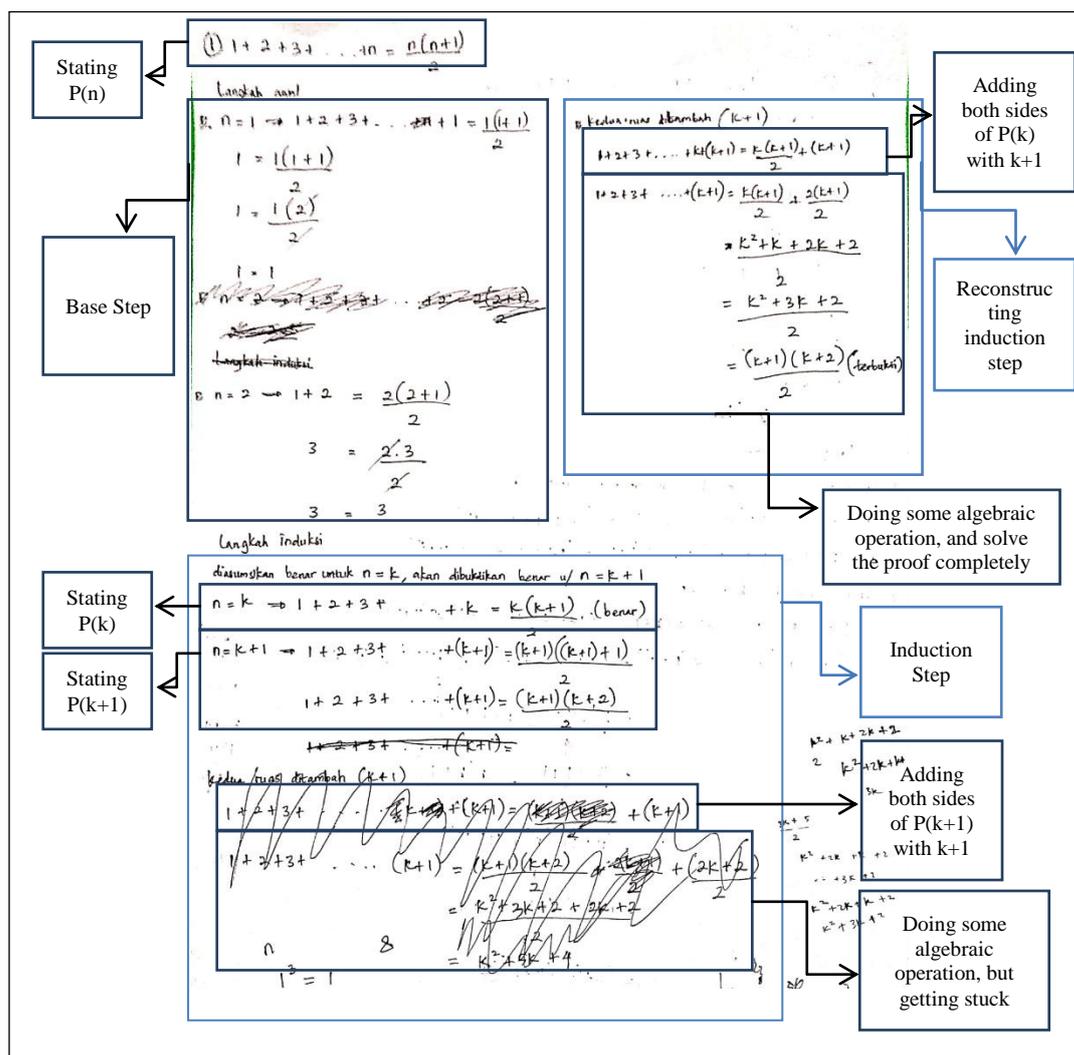


Figure 1. S1's Proving for the First Problem

S1 was interested in proving it because she was familiar with this formula. She often encountered it, but she has never proven it. That is why she was very interested in proving it. Based on the ATMI questionnaire, it was noted that S1 had considerable motivation in mathematics. She felt that the mathematics problems were quite interesting.

R : "Do you think that this problem is interesting?"

S1 : "Yes, it is interesting, but when it got to the algebra it was a bit confusing."

S1 continued to devise a plan in constructing the proof. She recalled information about the proof method by concerning the characteristics of what to be proven, namely proof using mathematical induction, as the problem is dealing with a natural number. She also mentioned the steps of the proof method, which includes the base and the induction step. Furthermore, S1 processed information by connecting this knowledge with the proof problem by stating  $P(n)$ .

$$P(n): 1 + 2 + 3 + \dots + n = \frac{n(k+1)}{2}$$

R : "Are you sure that this problem of proof can be solved using mathematical induction?"

S1 : "Yes, sure."

S1 had a high enough self-confidence. She felt confident in her plans because this problem is familiar to her. She encountered the same problem quite often. She also felt comfortable expressing her ideas to solve this proof problem. Mazana, Montero, and Casmir (2019) argue that students with high self-confidence believe in their abilities to be successful in learning mathematics, thus overcoming the fear of failing.

At the stage of carrying the plan out, S1 recalled information by explaining how to compile the base step. "In compiling the base step, we only need to substitute the value of  $n = 1$  to the equation" she said. S1 then processed this information by applying it directly to the equation, and she found equality.

So, S1 concluded that the base step was proven to be true. Since the base step was proven to be true, S1 continues to the induction step. S1 recalled information about how to construct the induction step, that is, by assuming  $P(k)$  is true, then  $P(k+1)$  is also true. Furthermore, she processed the information by stating  $P(k)$  and  $P(k+1)$  and arranging the induction step according to the steps mentioned earlier.

$$P(k): 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

$$P(k+1): 1 + 2 + 3 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}$$

S1 : Assume that the formula holds for  $n=k$ , it means that  $1 + 2 + 3 + \dots + k =$

$\frac{k(k+1)}{2}$  is true. From this statement I will prove that it holds as well for  $n=k+1$ , that

$$is\ 1 + 2 + 3 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}$$

Next, S1 added up the two segments of  $P(k+1)$  with  $k+1$ . She did some algebraic operations on the equation, but then she got stuck because the equation she got was not suitable for what she needed. So, she checked again the process she had done, and she realized that there was an error in it. What she should add with  $k+1$  are the two sides of  $P(k)$ , not  $P(k+1)$ .

S1 repeated the induction step by adding both sides of  $P(k)$  with  $k+1$ , equating the denominator, performing the algebraic operation on the right-hand side, and factorizing the

numerator. She found that this form gotten is the same as  $P(k + 1)$ . This means that  $P(k + 1)$  is true. Finally, she can conclude that  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  is proven to be true.

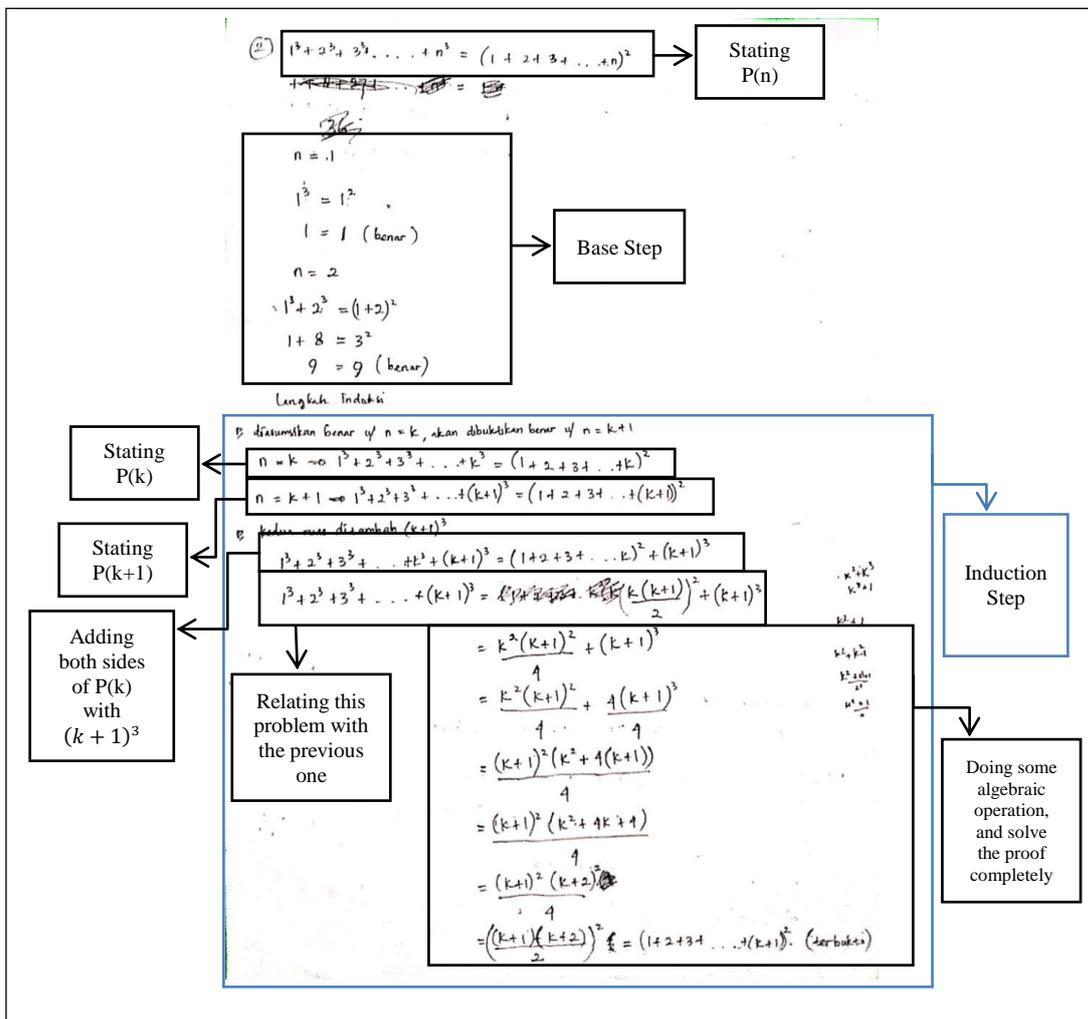


Figure 2. S1's Proving for the Second Problem

At the looking-back stage, S1 processed the information she had written. She checked every step she did if there is something wrong, and checked if the proof she constructed was logical and following her previous knowledge. She also rechecked the truth of some crucial parts in proving by mathematical induction, such as algebraic operation in induction step. She found that there is no error in it, and she concluded surely that the formula is proven to be true.

On the second problem, S1 began the proof by understanding what she needs to prove. At this stage, S1 received information by reading the task sheet and stored the information by reading the problem repeatedly and paraphrasing in her own language what she needed to prove. Then she recalled information in her memory if she had encountered a similar problem. She stated that she had ever encountered a similar problem, namely proving the series formula in the chapters on mathematical induction. Then she processed information by associating the information in the sheet task with the information she remembered, and she tried to state the statement in a mathematical expression.

In stating the statement in a mathematical expression, she had difficulty because she usually encounters proof in a mathematical sentence. She is sometimes confused in differing the words 'sum of the squares' and 'squares of the sum'. She devised a strategy to distinguish which one to prove,

$$1^3 + 2^3 + 3^3 + \dots + n^3 = 1^2 + 2^2 + 3^2 + \dots + n^2 \quad (1)$$

or

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2 \quad (2)$$

S1 wrote down the two formulas, read the problem slowly, and matched them with the two equations. Then she assumed that what meant by the problem in equation (2). To make her even more sure, she substituted  $n = 1, 2, 3$  to the equation, and she found equality. So S1 decided surely that the expression to be proven is equation (2).

After deciding which formula to prove, S1 continued to devise a plan. At this stage, she processed the information obtained from the task to determine the proof method by concerning the characteristics of this problem of proof. She decided to prove the problem using mathematical induction because this proof related to natural numbers.

*R* : "What is your reason for proving using mathematical induction? Is there a unique feature?"

*S1* : "Yes, there is a special feature. What we want to prove related to natural numbers. My teacher said that if it is related to natural numbers, the proof method is mathematical induction."

S1 also succeeded in retrieving information about the steps of the proof method by mathematical induction, which includes the base and the induction step. Then S1 processes the information by connecting it with the proof problem by stating  $P(n)$ .

$$P(n): 1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

At the stage of carrying out the plan, S1 recalled information about compiling the base step and explaining it. "In compiling the base step, we only need to substitute the value of  $n = 1$  to  $P(n)$ ," she said. Furthermore, she processed the information by applying it directly to the equation and she found equality. So the base step was proven to be true.

Then she continued to the induction step. She recalled information about how to construct the induction step; assuming  $P(k)$  is true, then  $P(k + 1)$  is true, followed by processing the information by stating  $P(k)$  and  $P(k + 1)$  and arranging the induction step according to the steps mentioned earlier.

$$P(k): 1^3 + 2^3 + 3^3 + \dots + k^3 = (1 + 2 + 3 + \dots + k)^2$$

$$P(k + 1): 1^3 + 2^3 + 3^3 + \dots + (k + 1)^3 = (1 + 2 + 3 + \dots + (k + 1))^2$$

In the induction step, S1 was having difficulties again. Based on her experience, in proving the formula for a series, she only needs to add the two sides of  $P(k)$  with the  $(k + 1)$ -th term. However, the right-hand side of  $P(k)$  is the square of a series, so it is difficult to expand. If she added the two sides  $P(k)$  with  $(k + 1)^3$ , then she got:

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3 = (1 + 2 + 3 + \dots + k)^2 + (k + 1)^3 \quad (3)$$

Meanwhile, what to be proven is  $P(k + 1)$ . S1 thought that it was not easy to bring (3) into  $P(k + 1)$ . She thought hard about what to do, read the problem over and over, and tried to remember something. Then she realized that the sum of  $k$  consecutive natural numbers in the sequence equals  $\frac{k(k+1)}{2}$ , as she had proved in the previous problem. She realized that changing the form of the series on the right-hand side would make it easier for her to perform algebraic operations. The form she got became

$$1^3 + 2^3 + 3^3 + \dots + (k + 1)^3 = \left(\frac{k(k + 1)}{2}\right)^2 + (k + 1)^3 \quad (4)$$

Initially, S1 did not know what she would do next with (4) to bring it to become  $P(k + 1)$ . After paying attention to what she needed to prove,  $P(k + 1)$ , she realized again that the right-hand side of  $P(k + 1)$  is the sum of  $k + 1$  consecutive natural numbers, equal to  $\frac{(k+1)(k+2)}{2}$ . So S1 needed to transform  $\left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3$  into  $\left(\frac{(k+1)(k+2)}{2}\right)^2$ . By doing some algebraic operations, finally, she could bring the form  $P(k)$  to  $P(k + 1)$ . So the problem was proven to be true.

In the induction step, many concepts of algebraic operations are used. Such as addition, multiplication, fraction, and algebraic factorization. S1 can remember and make good use of these concepts.

At the looking-back stage, S1 processed the information she had written. She checked every step she did if there is something wrong, and checked if the proof she constructed was logical and following her previous knowledge. She also rechecked the truth of some crucial parts in proving by mathematical induction, such as algebraic operation in induction step. She found that there is no error in it, and she concluded surely that the formula was proven to be true.

### Thinking Process of Student with Negative Attitude toward Mathematics (S2)

On the first problem, S2 began the proof by understanding what she needs to prove. At this stage, she received information by reading the task sheet and storing the information by paraphrasing in her language what she had to do and then recalling information in her memory if she had encountered a similar problem. She stated that she had ever encountered a similar problem, namely proving a series formula in one of the chapters she studied: the chapter on mathematical induction. Then she processed the information in the sheet task by associating it with the information she remembered. Finally, she understood that what she needs to do was proving the formula. S2 did not have difficulty understanding this problem, but this problem of proof did not interest her. She felt immediately depressed because she had to work on this proof.

- R : "Do you think that this problem is interesting?"  
S2 : "Oh, no. It is not interesting at all."

R : "Do you feel panic or depressed when doing this proving?"  
 S2 : "Yes, I am."

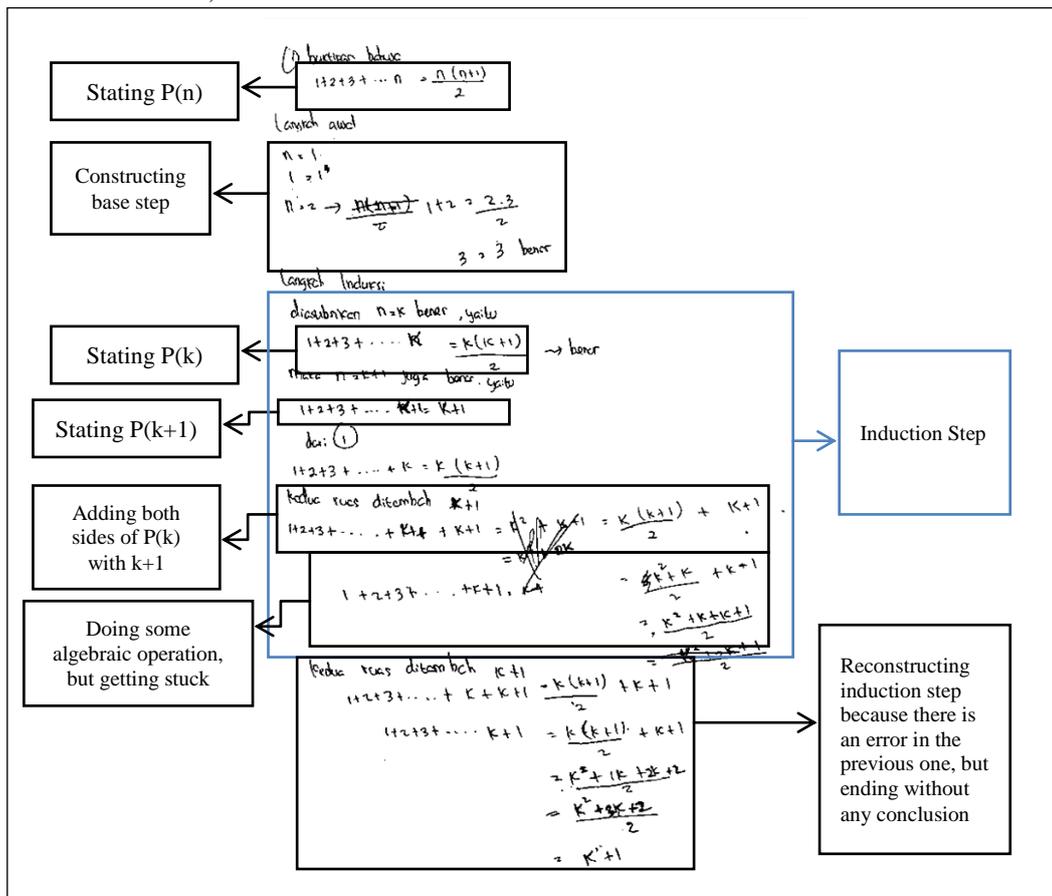


Figure 3. S2's Proving of the First Problem

S2 continued to devise a plan to construct the proof. She recalled information about the proving method by concerning the characteristics of what to be proven which is using mathematical induction since the problem is dealing with a natural number and mentioned the steps of the proving method, which includes the base and the induction step. She felt quite confident about the strategy. Furthermore, S2 connected her knowledge with the proof problem by stating  $P(n)$ .

$$P(n): 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

At the stage of carrying the plan out, S2 recalled information by explaining how to construct the base step. In constructing the base step, she substituted the value of  $n = 1$  to the equation. Then she processed it by doing some numerical operations and she found equality.

S2 also checked the mathematical truth for  $n = 2$ , and she obtained the result that the statement is true. From the process she did, she received information that the base step was proven to be true. Since the base step was proven to be true, she continued to the induction step. She recalled information about how to construct the induction step: assuming  $P(k)$  is true, then  $P(k+1)$  is also true. Then she processed the information by stating  $P(k)$  and  $P(k+1)$  and

constructing the induction step according to the steps mentioned earlier. She succeeded in stating  $P(k)$  correctly, but she could not express  $P(k + 1)$  correctly.

$$P(k): 1 + 2 + 3 + \dots + k = \frac{k(k + 1)}{2}$$

$$P(k + 1): 1 + 2 + 3 + \dots + (k + 1) = k + 1$$

She remembered  $P(k + 1)$  was obtained by replacing  $n$  in  $P(n)$  with  $k + 1$ . However, what she did is replacing the right-hand side of  $P(n)$  with  $k + 1$ . Also, S2 repeatedly said that she was not good at mathematics and could not solve this problem. This suggestion causes S2 cannot think clearly when doing mathematics.

Then S2 added the two sides of  $P(k)$  with the  $(k + 1)$ -th term, that is  $k + 1$ .

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{k(k + 1)}{2} + k + 1$$

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{k^2 + k}{2} + k + 1 \quad (5)$$

When equating the denominator on the right-hand side of (5), she did not multiply  $k + 1$  by 2

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{k^2 + k + k + 1}{2}$$

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{k^2 + 2k + 1}{2} \quad (6)$$

Since (6) did not match  $P(k + 1)$ , S2 looked back to the process. But she only checked the algebraic operation in the induction step instead of checking the whole process from the beginning. She was sure that there must be something wrong. She thought that the right-hand side of (6) should be  $k + 1$ , as in  $P(k + 1)$ . She found that she had a mistake when adding  $\frac{k^2+k}{2}$  with  $k + 1$  in (5) because  $k + 1$  should be multiplied by 2. Hence, she started the induction step again by adding the two sides of  $P(k)$  with  $k + 1$ . She did some algebraic operation, and she got this form

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{k^2 + 3k + 2}{2} \quad (7)$$

She got stuck again because she knew that this formula is not equivalent to  $P(k + 1)$ . She was sure that  $P(n)$  was true without proving it, but she did not know how to prove that if  $P(k)$  is assumed to be true, then  $P(k + 1)$  is true. Because she felt tired and desperate, she immediately wrote down

$$1 + 2 + 3 + \dots + k + (k + 1) = k + 1 \quad (8)$$

although she knew that (7)  $\neq$  (8). She did this because she did not want the proving process to stop halfway as well as did not want to linger with this problem of proof.

S2 : *"I don't know. I am confused. I know from the beginning that I will not solve this proving well. Whatever I do there must be a mistake."*

S2 then looked back to her process from the beginning, and she realized that the  $P(k + 1)$  she wrote was wrong, but she did not want to try again because she felt hopeless and was sure that she would fail. In fact, what S2 has written was correct. We can write (7) as

$$1 + 2 + 3 + \dots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2} \quad (9)$$

where (9) is the correct form of  $P(k + 1)$ . Because the  $P(k + 1)$  she wrote was not correct, she thought that (9) was wrong. If the  $P(k + 1)$  she stated was true, she would have possibly solved the proof completely.

On the second problem, S2 began the proof by understanding what she needed to prove. At this stage, she received information by reading the task sheet and storing it by paraphrasing what she had to do in her language. After reading this problem, she said that this problem of proof did not interest her at all. At this stage, S2 recalled information in her memory if she had encountered a similar problem. She knew that the problem in the task is a proof problem, but she had to translate it into mathematical expression first. Then she processed the information in the sheet task by associating the problem with her knowledge to find a mathematical expression with respect to the given information on the problem. Because she felt under pressure, it implies that the recalling and processing of information are not going well. In recalling and processing information, S2 has difficulty because the proof she usually encounters is already in a mathematical expression. However, she was still trying to solve this problem. Then she stated what to be proved in mathematical expression as follows.

$$1^3 + 2^3 + 3^3 + \dots + n^3 = 1^2 + 2^2 + 3^2 + \dots + n^2$$

She was sure that the equation to prove is that equation, so she began to devise a plan.

R : "Are you sure that this expression is what meant by this problem?"

S2 : "Yes, I am."

When devising a proof plan, S2 processed the information in the task sheet by considering the characteristics of this problem of proof. So, she recalled information regarding the proof method that suitable for the characteristic of the problem. She decided confidently to prove the problem using mathematical induction because this problem related to natural numbers. S2 also succeeded in retrieving information about the steps of the proof method using mathematical induction, which includes the base and the induction step. Then S2 processes the information by relating this problem to her knowledge by stating  $P(n)$ .

$$P(n): 1^3 + 2^3 + 3^3 + \dots + n^3 = 1^2 + 2^2 + 3^2 + \dots + n^2$$

At the stage of carrying out proving, S2 recalled the information by explaining how to construct the base and the induction step. In compiling the base step, she stated that she only need to substitute the value of  $n = 1$  to  $P(n)$ . She processed the information by applying it directly to the equation and she found equality. So the base step was proven to be true. Then S2 continued to the induction step. She recalled information about how to construct an induction step; assuming

$P(k)$  is correct, and she will show that  $P(k + 1)$  is also correct. Then she processed the information by stating  $P(k)$  and  $P(k + 1)$  and arranging the induction step according to the steps mentioned earlier.

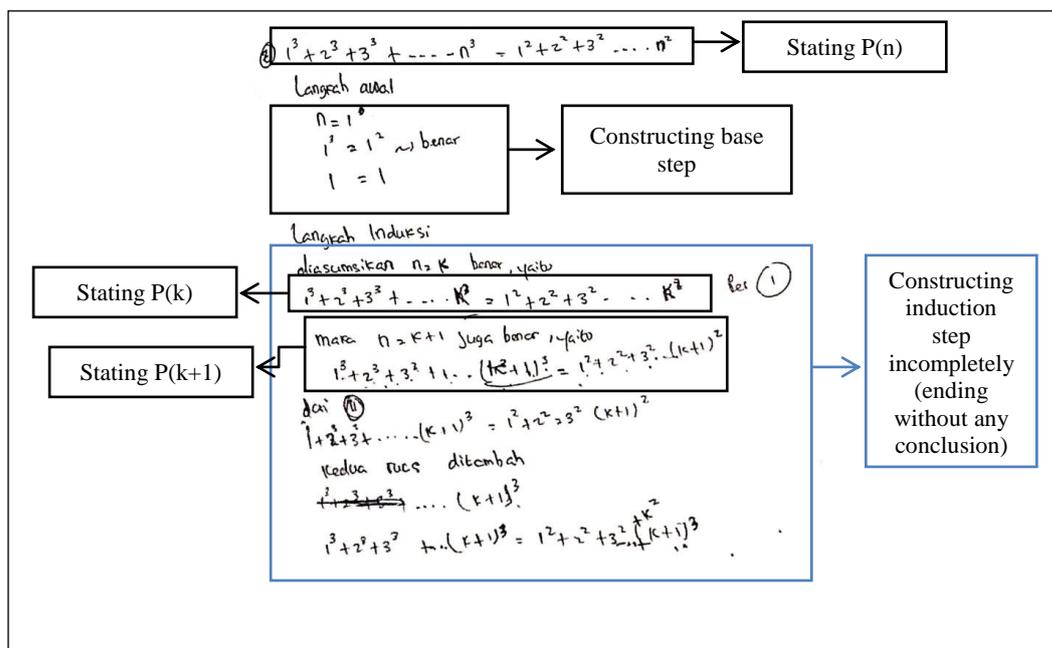


Figure 4. S2's Proving of the Second Problem

In the induction step, S2 was having difficulty. She could state  $P(k)$  and  $P(k + 1)$  correctly according to the equation she wrote earlier, but she did not know how to bring  $P(k)$  into  $P(k + 1)$ . She tried to add both sides of  $P(k)$  with  $(k + 1)^3$ , but she still did not know what next step she needs to do. Then she looked back to her process, and she doubted whether the  $P(n)$  was true or not. She substituted  $n = 2$  to the formula and found that it was wrong. So, the  $P(n)$  she wrote is wrong. Then she gave up and did not try to solve this problem anymore because she had already felt tough in the first problem and could not find a way out, so she was sure that she would experience the same thing in this problem.

Based on the description above, we can see the different process of the student with positive and negative toward mathematics in proving by mathematical induction. The student with a positive attitude toward mathematics tends to be more successful in constructing the proof. The statement is in line with the finding of Marchiş (2013) that attitudes towards mathematics are positively correlated with the ability to solve mathematical problems. The attitude affects the success in constructing proof and, of course, affects the thinking process while constructing it.

The student with a positive attitude toward mathematics thinks that the two proof problems are interesting. She always feels challenged when dealing with mathematical problems, so that she is motivated to solve them even though they are difficult. She also does not give up easily when dealing with difficulties, and she always tries to correct her mistakes when she finds them. Strong motivation to do mathematics makes her able to get good mathematics results (Tella, 2007).

In contrast, the student with a negative attitude toward mathematics felt pressured because she had to work on the proof problem. It also noted in the questionnaire that she could not think clearly when working on mathematics problems. This feeling indicates mathematics anxiety. [Akin and Kurbanoglu \(2011\)](#) state that mathematics anxiety is negatively related to attitude toward mathematics. It makes her having difficulty in processing information. It can be seen when compiling the induction step in the first problem. She could remember how to express  $P(k + 1)$ , but she could not process the information well, so what she does is not match what she says. As a result, the  $P(k + 1)$  she wrote was wrong. In line with what [Jones \(2007\)](#) stated, mathematics anxiety disrupts the ongoing task-relevant activities of working memory, slowing down performance and degrading its accuracy.

Also, it was noted that when working on proving, S2 always said that she would not be able to finish the proof properly no matter what she did. So she did not want to linger with the proofs and gave up quickly. This statement is negative self-judgment which is considered as one of the characteristics of a negative attitude towards mathematics ([Marchis, 2015](#)). Negative self-judgment shows that S2 has low self-confidence. She does not believe in her ability to be successful in constructing the proof. As mentioned in [Çiftçi & Yildiz \(2019\)](#), individuals with low self-confidence may avoid complex tasks, perceiving them as threats and giving up quickly when they encounter difficulties.

When the student with a negative attitude toward mathematics found her error in constructing the induction step at the first problem, she still tries to fix it. It shows a positive attitude toward mathematics. Although she has a negative attitude toward mathematics, it does not mean that she has a totally negative attitude. Sometimes, she also has a positive attitude because attitudes sometimes can change dramatically in a relatively short time ([Hannula, 2002](#)). However, her effort to fix her error is not successful. She still gets stuck and cannot solve the proof completely. Her unsuccessful effort in the first problem causes her to easily give up on the second problem, as [Hannula \(2002\)](#) stated that one's experience affects his/her attitude. She is sure that she will not complete the proof as to the first problem. The low self-confidence makes her easily give up before she even tried to construct the induction step.

In constructing proof by mathematical induction, difficulties are encountered not only by the student who has a negative attitude toward mathematics ([Çiftçi & Yildiz, 2019](#)). The student who has a positive attitude toward mathematics also experiences some difficulties, but she can respond positively. She faced these difficulties with confidence and did not give up easily. [Çiftçi & Yildiz \(2019\)](#) mention that individual with high self-confidence typically make more tremendous efforts to complete a task and are more persistent in the face of challenges. Hence they tend to be more successful in mathematics.

At the stage of understanding the second problem, it shows that a student with a positive attitude toward mathematics has her strategy in resolving her difficulties, as written in the

questionnaire that she is comfortable expressing her ideas on how to look for solutions to a complex problem in mathematics. She also enjoyed studying and doing mathematics. The more students enjoy mathematics, the more they are likely to engage in problem-solving, thus enhancing their learning and performance (Mazana et al., 2019). When arranging the induction step in the second problem, she reencountered a difficulty. She finds it difficult if she has to deal with squaring the series. But she did not give up. She kept thinking and trying to remember various things. In the end, she can solve the problem by relating it to the series formula in the first problem. This positive attitude made her able to complete the proof thoroughly.

## **CONCLUSION**

The student with a positive attitude toward mathematics can construct the proof of the two problems completely, while the student with a negative attitude toward mathematics cannot construct both of them completely. They both construct the proofs by understanding the proof problem, devising a plan, carrying out the plan, and looking back. But the process is not linear, there is some repetition in some stage. In each stage, they receive, store, recall, and process information relating to the proof problems. They both can recall information about the characteristics and the procedure of proof by mathematical induction. However, the student who has a negative attitude toward mathematics cannot process the information because she is too panic and feel depressed when constructing the proof. Hence, the processing of the information does not go well. She also feels pessimistic and unsure of her abilities, so she quickly gave up when dealing with some difficulties. In contrast, when the student who has a positive attitude towards mathematics finds some difficulties, she always tries to find a way out by expressing her ideas comfortably without feeling pressured. When she finds an error in her work, she always tries to fix it until she solves the proof completely.

From this study we can see that sometimes the failure of students in constructing proof is not because they do not have knowledge about it, but because of their negative attitude towards mathematics. For this reason, it is important for teachers to make students have positive attitude toward mathematics. Teaching mathematics in a fun way to make them no longer think that mathematics is something scary and do not feel under pressure when working with mathematics, giving appreciation to motivate them, and using real world problem to show that mathematics is useful in our daily life are some ways to make students have more positive attitude toward mathematics.

This study is still limited in constructing mathematical induction proof of female students with high mathematical ability. For further research, we recommend other researchers to study how the thinking process of students with different attitudes toward mathematics combined with another aspects, such as with different genders or different mathematical abilities, in another material.

## REFERENCES

- Adinata, Y. C., Budiyono, B., & Indriati, D. (2020). Misconceptions in mathematical induction for eleventh-grade students – HOTS categorized items Misconceptions in mathematical induction for eleventh-grade students – HOTS categorized items. *J. Phys.: Conf. Ser.* 1469 012153. <https://doi.org/10.1088/1742-6596/1469/1/012153>
- Ahmadi, Y., Kusumah, Y. S., & Jupri, A. (2019). Analysis of high school students' errors in mathematical proving : the case of mathematical induction. *J. Phys.: Conf. Ser.* 1211 012068, 1–7. <https://doi.org/10.1088/1742-6596/1211/1/012068>
- Akin, A., & Kurbanoglu, I. N. (2011). The Relationship between math anxiety, math attitudes, and self-efficacy: A structural equation model. *Studia Psychologica*, 53(3), 263–273. <https://arts.units.it/handle/11368/2942931>
- Ashkenazi, Y., & Itzkovitch, E. (2014). Proof by mathematical induction. *International Journal of Innovation and Research in Educational Sciences*, 1(3), 186–190. [http://www.ijires.org/administrator/components/com\\_jresearch/files/publications/IJIRES\\_86\\_Final.pdf](http://www.ijires.org/administrator/components/com_jresearch/files/publications/IJIRES_86_Final.pdf)
- Çiftçi, K., & Yildiz, P. (2019). The effect of self-confidence on mathematics achievement: The meta-analysis of Trends in International Mathematics and Science Study (TIMSS). *International Journal of Instruction*, 12(2), 683–694. <https://doi.org/10.29333/iji.2019.12243a>
- Cooper, A. A. (2019). Understanding mathematical induction by writing analogies. *Prompt: A Journal of Academic Writing Assignment*, 3(2), 14–24. <http://thepromptjournal.com/index.php/prompt/article/view/38>
- Cyr, S. (2011). Development of beginning skills in proving and proof writing by elementary school students. *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education*, 1–10. [http://lettredelapreuve.org/pdf/CERME7/Cerme7\\_WG1\\_Cyr.pdf](http://lettredelapreuve.org/pdf/CERME7/Cerme7_WG1_Cyr.pdf)
- Doruk, M. (2019). Preservice mathematics teachers' determination skills of proof techniques: The case of integers. *International Journal of Education in Mathematics Science and Technology*, 7(4), 335–348. <https://www.ijemst.net/index.php/ijemst/article/view/729>
- Fauziyah, N. (2020). *Proses kognisi siswa Autism Spectrum Disorder (ASD) tingkat SMA dalam menyelesaikan masalah matematika berdasarkan perbedaan tingkat inteligensi*. Universitas Negeri Surabaya.
- Furinghetti, F., & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way : affective and cognitive factors in proving. *Educ Stud Math*, 70, 71–90. <https://doi.org/10.1007/s10649-008-9134-4>
- Güler, G. (2016). The difficulties experienced in teaching proof to prospective mathematics teachers: Academician views. *Higher Education Studies*, 6(1), 145–158. <https://doi.org/10.5539/hes.v6n1p145>
- Hannula, M. S. (2002). Attitude towards mathematics: Emotions, expectations and values. *Educational Studies in Mathematics*, 49(1), 25–46. <https://link.springer.com/article/10.1023/A:1016048823497>
- Hasan, B. (2016). Proses berpikir mahasiswa dalam mengkonstruksi bukti menggunakan induksi matematika berdasarkan teori pemrosesan informasi. *APOTEMA : Jurnal Program Studi Pendidikan Matematika*, 2(1), 33–40. <https://doi.org/10.31597/ja.v2i1.126>
- Hemmi, K., & Löfwall, C. (2010). Why do we need proof. *CERME 6, Congress of the European Society for Research in Mathematics Education*, 201–210. <http://www.divaportal.org/smash/record.jsf?pid=diva2:400484>
- Isroil, A., Budayasa, I. K., & Masriyah, M. (2017). Profil berpikir siswa SMP dalam menyelesaikan

- masalah matematika ditinjau dari kemampuan matematika. *Jurnal Review Pembelajaran Matematika*, 2(2), 93–105. <https://doi.org/10.15642/jrpm.2017.2.2.93-105>
- Jones, V. O. (2007). Cognitive processes during problem solving of middle school students with different levels of mathematics anxiety and self-esteem: Case studies. *Dissertation Abstracts International Section A: Humanities and Social Sciences Vol*, 67(8-A), 2915.
- Lim, S. Y., & Chapman, E. (2013). Development of a short form of the attitudes toward mathematics inventory. *Educ Stud Math*, 82, 145–164. <https://doi.org/10.1007/s10649-012-9414-x>
- Marchis, I. (2015). Factors that influence secondary school students' attitude to mathematics. *Procedia - Social and Behavioral Sciences*, 29(October), 786–793. <https://doi.org/10.1016/j.sbspro.2011.11.306>
- Marchiş, I. (2013). Relation between students' attitude towards mathematics and their problem solving skills. *PedActa*, 3(2), 59–66.
- Martino, P. Di, & Zan, R. (1992). Attitude toward mathematics: Some theoretical issues. *Dipartimento Di Matematica Università Di Pisa, ITALY*, 1–8. <https://erepo.uef.fi/handle/123456789/7398>
- Mazana, M. Y., Montero, C. S., & Casmir, R. O. (2019). Investigating students' attitude towards learning mathematics. *International Electronic Journal of Mathematics Education*, 14(1), 207–231.
- Mudhiah, I. D., & Amin, S. M. (2020). Profil berpikir siswa dalam menyelesaikan soal matematika ditinjau dari kecerdasan emosional. *MATHEdunesa: Jurnal Ilmiah Pendidikan Matematika*, 9(1), 136–144.
- Ngilawajan, D. A. (2013). Proses berpikir siswa SMA dalam field independent dan field dependent. *Pedagogia*, 2(1), 71–83. <http://ojs.umsida.ac.id/index.php/pedagogia/article/view/48>
- Pantaleon, K. V., Juniati, D., Lukito, A., & Mandur, K. (2018). The written mathematical communication profile of prospective math teacher in mathematical proving The written mathematical communication profile of prospective math teacher in mathematical proving. *Journal of Physics: Conference Series*, 947(1), 012070. <https://doi.org/10.1088/1742-6596/947/1/012070>
- Relaford-doyle, J. (2020). Characterizing students' conceptual difficulties with mathematical induction using visual proofs. *International Journal of Research in Undergraduate Mathematics Education*, 1–20.
- Tapia, M. (1996). The attitudes toward mathematics instrument introduction. *Annual Meeting of the Mid-South Educational Research Association*, 1–21. <https://eric.ed.gov/?id=ED404165>
- Tella, A. (2007). The impact of motivation on student's academic achievement and learning outcomes in mathematics among secondary school students in Nigeria. *Eurasia Journal of Mathematics, Science and Technology Education*, 3(2), 149–156. <https://doi.org/10.12973/ejmste/75390>
- Warli, Cintamulya, I., & Rahayu, P. (2020). Scaffolding process based on students diagnostic difficulties in proving group problems by using mathematics mapping Scaffolding process based on students diagnostic difficulties in proving group problems by using mathematics mapping. *Journal of Physics: Conference Series*, 1422(1), 012012. <https://doi.org/10.1088/1742-6596/1422/1/012012>
- Widyastuti, R. (2015). Proses berfikir siswa dalam menyelesaikan masalah matematika berdasarkan teori Polya ditinjau dari adversity quotient tipe climber. *Al-Jabar : Jurnal Pendidikan Matematika*, 6(2), 183–194. <http://ejournal.radenintan.ac.id/index.php/al-jabar/article/view/48/0>